

# AN INTEGRAL SOLUTION FOR THE MELTING OF A FINITE SLAB WITH CHF HEATING AND RADIATION COOLING

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## Abstract

Integral method was employed to obtain approximate solution for the melting process of a finite slab which is subjected to constant heat flux heating to one surface and radiation cooling to the opposite surface. Locations of the interface between liquid and solid phases were determined. Results obtained in this paper would be useful for the design of thermal control system in space vehicles.

## Introduction

Thermal control technology plays an important role during the development of space flight. The principle of solid-liquid phase change can be effectively used to cool the periodically operating electronic components. A schematic diagram of the thermal control system with PCM (phase change materials) is shown in Fig.1. Heat dissipated by the chips will be absorbed by the PCM via the latent heat of fusion during the operating period and the melted PCM will be re-solidified due to the removal of heat through the space radiator during the non-operating period of the electronic equipment. Owing to the fixed fusion point of PCM, the temperature of the electronic components can be maintained at a certain average level even though the actual amount of dissipating heat fluctuates. Thus, the surface area of a space-radiator can be calculated on the basis of average dissipating power and the surface area of the space-radiator can be significantly reduced<sup>(1)</sup>

Radiation cooling becomes the only mode of heat removal during the operation of devices beyond the earth's atmosphere where convection vanishes. The present authors solved the solidification process of PCM in a semi-infinite region subjected to radiation cooling<sup>(2)</sup>, which corresponds to the situation during the non-operating period of the electronic equipment. During the operation period of the electronic components, the heat transfer process of the thermal control system can be simplified as the melting process of a finite PCM slab with one surface subjected to constant heat flux heating and the other to radiation cooling. In order to get a whole picture of the performance of electronic equipment cooled with PCM, the above mentioned problem should be solved.

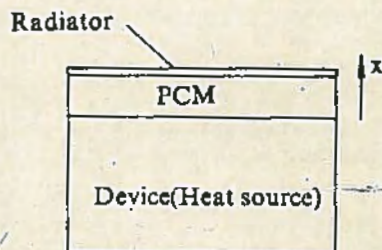


Fig.1 Schematic diagram of the thermal control system with PCM.

Using the perturbation method, Goodman solved the melting problem of a one dimensional finite slab with constant flux heating to one wall and the opposite wall kept at constant temperature or adiabatic<sup>(3)</sup>. But to the best of our knowledge, there is no published results of the melting



process of PCM subjected to constant flux heating to one wall and radiation cooling to the opposite wall. The present paper will aim at solving this problem. The locations of the solid-liquid interface with time will also be discussed.

### Mathematical Descriptions

A simplification used is the assumption of a uniform temperature equals to the fusion point of the PCM in the solid just before the operation of electronic equipment begins ( $t=0$ ). As soon as  $t>0$ , heat generated by the chips begins to dissipate and the PCM just adjacent to the heated wall begins to melt. At the same time, the space radiator begins to deliver heat into the space and the temperature of the PCM at the radiator wall decreases. Therefore, there exists a temperature gradient in both the liquid and solid phases and both of the temperature profiles need to be solved, i.e. the problem is a so-called double region problem. Its mathematical descriptions can be expressed as follows:

Liquid phase region

$$\frac{\partial^2 T_1}{\partial x^2} = \frac{1}{a_1} \frac{\partial T_1}{\partial t}, \quad 0 < x < s(t), \quad t > 0 \quad (1)$$

$$\frac{\partial T_1}{\partial x} = -\frac{1}{k_1} q, \quad x = 0, \quad t > 0 \quad (2)$$

Solid phase region

$$\frac{\partial^2 T_2}{\partial x^2} = \frac{1}{a_2} \frac{\partial T_2}{\partial t}, \quad s(t) < x < W, \quad t > 0 \quad (3)$$

$$\frac{\partial T_2}{\partial x} = -\frac{1}{k_2} \epsilon \sigma T_2^4, \quad x = W, \quad t > 0 \quad (4)$$

$$T(x, t) = T_m, \quad 0 < x < W, \quad t = 0 \quad (5)$$

Solid-liquid interface

$$T_1(x, t) = T_2(x, t) = T_m, \quad x = s(t), \quad t > 0 \quad (6)$$

$$k_2 \frac{\partial T_2}{\partial x} - k_1 \frac{\partial T_1}{\partial x} = \rho L \frac{ds}{dt}, \quad x = s(t), \quad t > 0 \quad (7)$$

Define the following dimensionless parameters:

$$\left. \begin{aligned} X &= \frac{x}{W}, \quad S = \frac{s}{W}, \quad \tau = \frac{a_1 t}{W^2}, \quad K_a = \frac{a_1}{a_2} \\ \theta_1 &= \frac{c_{p1}(T_1 - T_m)}{L}, \quad \theta_2 = \frac{c_{p2}(T_2 - T_m)}{L} \\ Ste &= \frac{c_{p1} q W}{k_1 L}, \quad S_c = \frac{c_{p2} (\epsilon \sigma T_m^4) W}{k_2 L} \end{aligned} \right\} \quad (8)$$

Then eqs.(1-7) take on the following form when they are written in the dimensionless parameters:

$$\frac{\partial^2 \theta_1}{\partial X^2} = \frac{\partial \theta_1}{\partial \tau}, \quad 0 < X < S(\tau), \quad \tau > 0 \quad (9)$$

$$\frac{\partial \theta_1}{\partial X} = -Ste, \quad X = 0, \quad \tau = 0 \quad (10)$$

$$\frac{\partial^2 \theta_2}{\partial X^2} = K_a \frac{\partial \theta_2}{\partial \tau}, \quad S(\tau) < X < 1, \quad \tau > 0 \quad (11)$$

$$\frac{\partial \theta_2}{\partial X} = -S_c \left(1 + \frac{L}{c_{p2} T_m} \theta_2\right)^4, \quad X = 1, \quad \tau > 0 \quad (12)$$

$$\theta_2(X, \tau) = 0, \quad 0 < X < 1, \quad \tau = 0 \quad (13)$$

$$\theta_1(X, \tau) = \theta_2(X, \tau) = 0, \quad X = S(\tau), \quad \tau > 0 \quad (14)$$

$$\frac{\partial \theta_2}{\partial X} - K_a \frac{\partial \theta_1}{\partial X} = K_a \frac{dS}{d\tau}, \quad X = S(\tau), \quad \tau > 0 \quad (15)$$

### Integral Approximate Solution

Since initially the temperature in the Solid PCM is assumed to be at its phase change temperature  $T_m$ , this greatly simplified the mathematical descriptions and enhances the use of integral method. The phase change ensues and the melt layer appears as soon as the electronic components begin to work. The thickness of the liquid layer becomes larger with time. At the same time, owing to the radiation cooling to the opposite surface of the PCM plate, a colder layer will be built up near the opposite surface and its thickness increases with time too. Evidently, as shown in Fig.2, the process can be treated as two independent parts before the boundaries of the liquid layer and the colder layer meet together. The whole process can be simplified as the melting process of a semi-infinite plate with constant



flux heating in the left part of the plate and the conduction process of a semi-infinite plate with radiation cooling in the right part of the plate. The temperature of the middle layer (i.e. between the boundaries of liquid layer and colder layer) remains unchanged. As time goes on, the boundary of the liquid layer meets together with that of the colder layer, the left part is no longer independent of the right part, thus the problem should be solved as a whole then. According to the above analysis, the problem will be solved as follows.

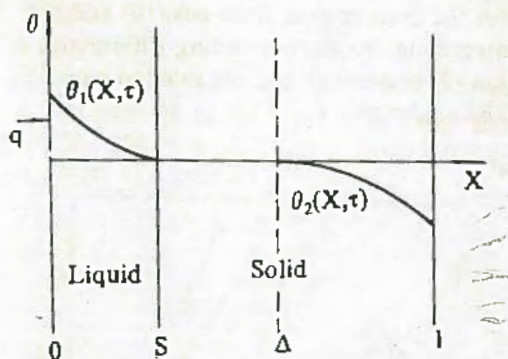


Fig.2 Temperature profile before the boundaries meet together

### 1. Solution before the boundaries meet together

As mentioned above, the left part corresponds to the melting process of a semi-infinite plate without initial subcooling. This problem has already been solved with the integral method<sup>(4)</sup> and the location of the interface between solid and liquid phases can be calculated from the following equation:

$$(SteS + 5 + \sqrt{1 + 4SteS})S = 6Ste \tau \quad (16)$$

and the temperature profile in the liquid phase region will be:

$$\theta_1(X, \tau) = \frac{Ste}{2S} (X - S)^2 - \frac{X^2 - S^2}{4S^2} (\sqrt{1 + 4SSSte} - 1) \quad (17)$$

The right part corresponds to the conduction process of a semi-infinite region exposed to

radiation cooling. Integrating the eq.(3) between the interval  $(1, 1-\Delta)$  and taking the definition of colder layer into account, one obtains the following integral equation:

$$\frac{\partial \theta_2}{\partial X} \Big|_{X=1} = K_o \frac{d\Theta_2}{d\tau} \quad (18)$$

Where:

$$\Theta_2 = \int_{\Delta}^1 \theta_2 dX \quad (19)$$

Let the temperature profile in the colder layer be represented by a quadratic expression. Two conditions are needed for evaluation of the two constants. These conditions are those shown by eq.(12) and the definition of colder layer. Then the temperature profile in colder layer becomes:

$$\theta_2 = \eta(\tau) \left( \frac{X - \Delta}{1 - \Delta} \right)^2 \quad (20)$$

in which,  $\eta(\tau)$  can be given by the following transcendental equation:

$$\frac{2K_o \eta^2}{S_c \left[ 1 + \frac{L}{c_{p2} T_m} \eta \right]^4} = \int_0^{\tau} 3S_c \left[ 1 + \frac{L}{c_{p2} T_m} \eta \right]^4 d\tau \quad (21)$$

thickness of the colder layer  $1-\Delta$  can be calculated from:

$$1 - \Delta = - \frac{2\eta}{S_c \left[ 1 + \frac{L}{c_{p2} T_m} \eta \right]^4} \quad (22)$$

Starting from  $\tau=0$ , one may obtain the location of the interface between solid and liquid phases and thickness of the colder layer as some function of time. Of course, the above method is valid only before the boundary of the liquid layer meets the boundary of the colder layer, i.e.  $\Delta(\tau) > S(\tau)$ . Once the two boundaries meet together, other method should be used which will be described shortly. Assume at time  $\tau_1$ , the two boundaries meet together, then  $\tau_1$  must satisfy the expression:



$$S(\tau_1) = \Delta(\tau_1) \quad (23)$$

## 2. Solution after the boundaries meet together

The location of the interface  $S_1$  when the boundaries of liquid layer and colder layer meet together is:

$$S_1 = S(\tau_1) = \Delta(\tau_1) \quad (24)$$

and the temperature profiles in the solid and liquid regions are:

$$\theta_2(X, \tau_1) = \eta(\tau_1) \left( \frac{X - S_1}{1 - S_1} \right)^2 \quad (25)$$

$$\theta_1(X, \tau) = \frac{Ste}{2S_1} (X - S_1)^2 - \frac{X^2 - S_1^2}{4S_1^2} (\sqrt{1 + 4S_1 Ste} - 1) \quad (26)$$

respectively. Firstly, considering the solid region, integrating both sides of the differential eq.(11) between the interval  $(S, 1)$  and taking the eq.(14) into account, one obtains the integral equation for solid phase:

$$\frac{\partial \theta_2}{\partial X} \Big|_{X=1} - \frac{\partial \theta_2}{\partial X} \Big|_{X=S} = K_s \frac{d\Theta}{d\tau} \quad (27)$$

where

$$\Theta_2 = \int_S^1 \theta_2 dX \quad (28)$$

The temperature profile in the solid region is assumed to be a quadratic function:

$$\theta_2 = b_1 \left( \frac{X - S}{1 - S} \right) + b_2 \left( \frac{X - S}{1 - S} \right)^2 \quad (29)$$

which identically satisfies eq.(14), then by substitution of eq.(29) into eq.(12), one obtains:

$$\frac{b_1 + 2b_2}{1 - S} = -S_c \left[ 1 + \frac{L}{c_p T_m} (b_1 + b_2) \right] \quad (30)$$

upon substitution eq.(29) into integral eq.(27) and integrating, it follows:

$$(1 - S)(3b_1 + 2b_2) = \int_{\tau_1}^{\tau} \frac{12b_2}{K_s(1 - S)} d\tau + 2\eta(\tau_1)(1 - S_1) \quad (31)$$

Evaluating  $b_1(\tau)$  and  $b_2(\tau)$  from eqs.(30) and (31), the temperature profile in the solid region can be obtained. The initial values of  $b_1(\tau)$  and  $b_2(\tau)$  is given by eq.(25) as:

$$b_1(\tau_1) = 0 \quad (32)$$

$$b_2(\tau_1) = \eta(\tau_1) \quad (33)$$

For the present case, from eqs.(10) and (14) and integrating the corresponding differential equation (9) between  $(0, S)$ , the integral equation for the liquid region is:

$$\frac{\partial \theta_1}{\partial X} \Big|_{X=S} + Ste = \frac{d\Theta_1}{d\tau} \quad (34)$$

where

$$\Theta_1 = \int_0^S \theta_1 dX \quad (35)$$

Once again let the temperature profile in the liquid region be represented by a quadratic function, its constants can be calculated by eqs.(10),(14) and (15), then the temperature profile for the liquid phase region can be written as:

$$\theta_1(X, \tau) = \frac{Ste}{2S} (X - S)^2 - \frac{p}{2S^2} (X^2 - S^2) \quad (36)$$

where

$$p = \frac{1}{2} \sqrt{\left[ 1 + \frac{b_1}{K_s(1 - S)} \right]^2 + 4SteS} - \frac{1}{2} \left[ 1 + \frac{b_1}{K_s(1 - S)} \right] \quad (37)$$

when  $\tau = \tau_1$ , eq.(36) reduces to eq.(26).

Substituting integral eqs.(27) and (34) to the interface eq.(15), one obtains:

$$\frac{dS}{d\tau} = Ste - \frac{d\Theta_1}{d\tau} - \frac{d\Theta_2}{d\tau} + \frac{b_1 + 2b_2}{K_s(1 - S)} \quad (38)$$



Integrating each side of the above equation between  $(\tau_1, \tau)$  and upon substitution eqs.(28), (35) into the integration results, the location of interface between the solid and liquid phases can be written as follows:

$$\begin{aligned} & \left[1 + \frac{1}{6}(SteS + 2p) - \left(\frac{b_1}{2} + \frac{b_2}{3}\right)S\right] \\ & = Ste\tau - \left(\frac{b_1}{2} + \frac{b_2}{3}\right) + \int_{\tau_1}^{\tau} \frac{b_1 + 2b_2}{K_s(1-S)} d\tau \\ & + \frac{1}{3}\eta(\tau_1)(1-S_1) \end{aligned} \quad (39)$$

Combining eqs.(30),(31) and (39), one finally obtains the location of the interface between solid and liquid phases after the liquid layer meets the colder layer as a function of time.

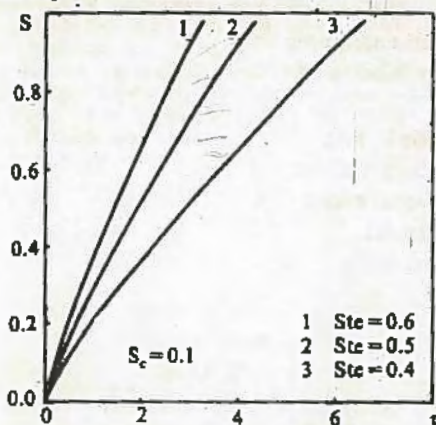


Fig.3 The history of interface location with different Ste's

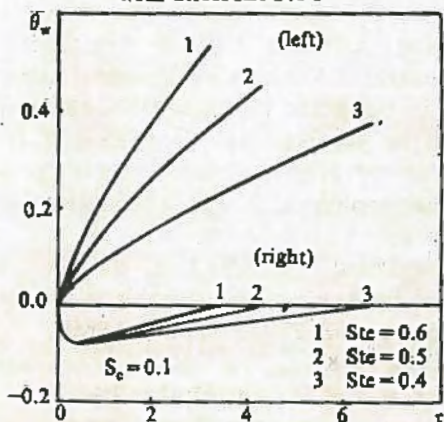


Fig.4 The history of surface temperature with different Ste's

## Results and Discussions

The melting process of octadecane was calculated and the results are shown in Fig.3-6.

The interface location-time history with different stefan numbers while  $S_c = 0.1$  is shown in Fig.3. Obviously, the melting rate increase as the Stefan number is increased. The temperature-time history of the surface  $X=0$  and  $X=1$  for the finite slab PCM with different Stefan numbers while  $S_c = 0.1$  is given in Fig.4. The temperature of the left surface i.e.  $X=0$  increases monotonically as melting goes on, and the larger the Stefan number, the higher the surface temperature will be. Take the largest Stefan number 0.6 as an example, the surface temperature at  $X=0$  reaches  $\theta_1 = 0.53$ , i.e.  $87.3^\circ\text{C}$  when

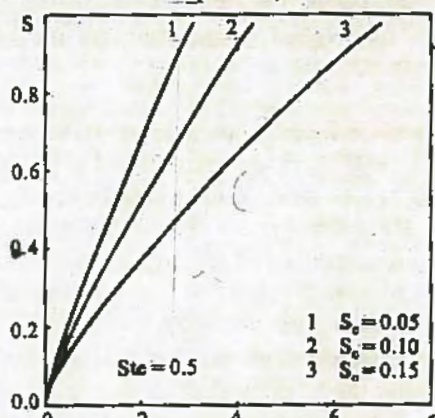


Fig.5 The history of interface location with different  $S_c$ 's

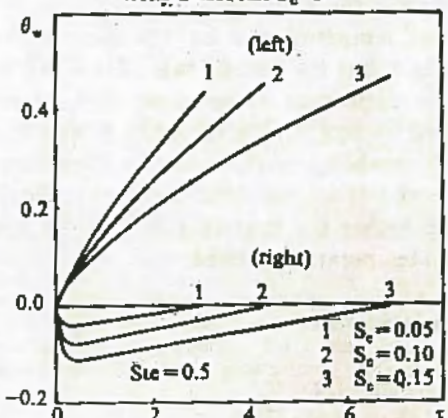


Fig.6 The history of surface temperature with different  $S_c$ 's



the entire volume of PCM is melted. It can be concluded that even though the PCM thermal control has rather large capability of absorbing heat, the low conductivity of the PCM leads to the relative high surface temperature. Therefore, high conductivity filler may be added into the PCM as a remedial measure. In addition, natural convection occurring in the melting PCM will reduce the temperature level, but it is not really the case in space. As to the temperature at  $X = 1$ , i.e. the space radiator surface temperature, it drops down rapidly at the very beginning of the process because of the radiation cooling occurs immediately after  $\tau > 0$ , but the influence of the heating on the left wall has not reached at  $X = 1$ . As time goes on, the influence reaches and the right wall temperature begins to rise and of course, the larger the Stefan number, the higher the temperature. Eventually, the right wall temperature returns to its original temperature i.e. the fusion temperature.

The interface location as a function of time at  $Ste = 0.5$  with  $S_c$  as the parameters is shown in Fig.5. It can be seen that the smaller the  $S_c$ , the higher the melting rate will be. The relationship between temperature of left and right surfaces of the PCM and time is given in Fig.6. Again,  $Ste = 0.5$  and different values of  $S_c$  were used to calculate these results. It is evident that the smaller the  $S_c$  the higher the left wall temperature will be. It is interesting to note that the final temperature of the left wall is almost the same despite the different values of  $S_c$ . As to the right wall temperature, it has the same tendency as in Fig.4, but the heating rate of the wall is almost the same even if the values of  $S_c$  vary. As indicated by eq.(12), the values of  $S_c$  implies the cooling capability of the radiator. Therefore, it is quite understandable that the weaker the cooling, the higher the melting rate and the higher the wall temperatures will be.

#### Nomenclature

$a$  thermal diffusivity  
 $b_1$  constant in eq.(29)  
 $b_2$  constant in eq.(29)

$c_p$  specific heat  
 $k$  thermal conductivity  
 $K_a$  thermal diffusivity ratio  
 $L$  latent heat of melting  
 $p$  defined in eq.(37)  
 $q$  surface heat flux  
 $s$  solid-liquid interface location  
 $S$  dimensionless interface location  
 $S_c$  subcooling parameter  
 $Ste$  Stefan number  
 $t$  time  
 $T$  temperature  
 $W$  width of the finite slab  
 $x$  coordinate  
 $X$  dimensionless coordinate  
 $\Delta$  dimensionless thickness of colder layer  
 $\varepsilon$  emissivity  
 $\theta$  dimensionless temperature  
 $\sigma$  Stefan-Boltzmann constant  
 $\tau$  dimensionless time  
 $\eta$  defined in eq.(20)

#### Subscripts

1 liquid phase  
 2 solid phase  
 i initial  
 m melting

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