1. INTRODUCTION

The lattice Boltzmann method (LBM) (Succi, 2001) has been developed to be a powerful numerical method for fluid flow simulation in the last three decades. Traditional computational fluid dynamic (CFD) methods, such as finite volume method (FVM) and finite difference method (FDM), solve difference mass and momentum equations to obtain the velocity, density, and pressure directly. Different from them, the LBM solves the mesoscopic density distribution parameters first and then uses them to calculate the macroscopic parameters that can also satisfy the mass and momentum equations. The LBM has been applied to many fluid flow and heat transfer problems, including incompressible fluid flow (Zou et al., 1995; Li et al., 2014d), natural convection (Li et al., 2016a,b; Rahmati et al., 2009), porous media fluid flow (Chen and Doolen, 1998; Guo and Zhao, 2002), and phase change problems (Chatterjee and Chakraborty, 2005; Li et al., 2014c, 2015b). Comparing with the traditional methods, the LBM shows its advantages in easy code settings, applicability in parallel computing (Wang and Aoki, 2011), and suitability to complex fluid flow. Several hybrid methods are developed to take advantages of both the CFD and LBM (Li et al., 2014a,b). The LBM can also be coupled with the Monte Carlo method to solve fluid flow and heat transfer problems (Li et al., 2015a).

Some basic problems still remain in this algorithm. For example, there are many boundary condition setting methods in the LBM without reaching agreement. Latt and Chopard (2008) compared five straight common boundary conditions and suggested that all the methods can yield macroscopic results of the same
accuracy for several cases. Collison and streaming are two basic processes in the LBM. Their boundary conditions fall into two categories: (1) recovering the unknown density distribution step after streaming step on the boundary, (2) replacing all density distributions. These methods yield different results when the boundary speed is not zero. In that case, the density distributions entering into the system differ from those leaving the system which violates the local mass balance (Yin et al., 2012).

Numerous efforts have been made to study the cases of nonzero velocity to discuss local mass balance for the LBM boundary. A curved boundary treatment was proposed (Mei et al., 1999, 2002). The no-slip curved boundary is approximated by a series of stairs. The velocities on the stairs obtained from the difference are not equal to zero. Correspondingly, it may violate the local mass balance. Bao et al. (2008) and Verschaeve and Muller (2010) applied different settings for the local mass balance on a curved boundary. The velocity on the moving boundary is also nonzero in the LBM. Coupanec and Verschaeve (2011) derived a mass conserving boundary condition for tangentially moving walls in the LBM. An enhanced mass conserving closure scheme was employed for the LBM hydrodynamics under open boundary conditions (Hollis et al., 2006). There are some different opinions on the local mass balance. It was argued that this local mass balance is fundamentally flawed leading incorrect pressure (Ladd and Verberg, 2001; Nguyen and Ladd, 2002).

Ginzbourg and d’Humieres (1996) demonstrated that the total mass balance can be reached even though the local mass balance is not satisfied in the LBM boundary conditions (Filippova and Hanel, 1998). Mass and momentum transfer across the curved boundary in the LBM are reviewed in Yin et al. (2012). They concluded that the adding of momentum to the density distribution reduction had no direct influence on flow and pressure fields, but the incorrect fluid–particle interaction might affect the results of simulation of particulate suspensions. Mass balance is the conservation law based on macroscopic parameters (velocity and density) and the local mass balance has no direct relation with the conservation law, since it is based on the mesoscopic density distribution parameter. In this paper, the LBM with three common boundary conditions [the Zou–He method (Zou and He, 1997), finite difference velocity gradient method, and regularized method] are employed to solve the fluid flow problem with the Dirichlet velocity boundary. All three methods cannot

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**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>lattice acceleration</td>
</tr>
<tr>
<td>$c$</td>
<td>lattice speed</td>
</tr>
<tr>
<td>$c_s$</td>
<td>lattice sound speed</td>
</tr>
<tr>
<td>$e_a$</td>
<td>velocity in every direction</td>
</tr>
<tr>
<td>$f$</td>
<td>density distribution</td>
</tr>
<tr>
<td>$I$</td>
<td>unit tensor</td>
</tr>
<tr>
<td>$K$</td>
<td>Chapman–Enskog expansion coefficient</td>
</tr>
<tr>
<td>$P$</td>
<td>nondimensional pressure</td>
</tr>
<tr>
<td>$Q$</td>
<td>lattice weight tensor</td>
</tr>
<tr>
<td>$r$</td>
<td>lattice location vector</td>
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<tr>
<td>$t$</td>
<td>lattice time</td>
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</tbody>
</table>

**Greek symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>horizontal velocity in lattice unit</td>
</tr>
<tr>
<td>$\nu$</td>
<td>vertical velocity in lattice unit</td>
</tr>
<tr>
<td>$U$</td>
<td>nondimensional horizontal velocity</td>
</tr>
<tr>
<td>$V$</td>
<td>nondimensional vertical velocity</td>
</tr>
<tr>
<td>$V$</td>
<td>velocity</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>stress tensor</td>
</tr>
<tr>
<td>$\tau$</td>
<td>relaxation time</td>
</tr>
<tr>
<td>$\omega$</td>
<td>lattice weight</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>collision operator</td>
</tr>
</tbody>
</table>

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*Heat Transfer Research*
satisfy the local mass balance for the nonzero boundary. These three boundary conditions will be discussed based on the mass conservation law directly.

2. PROBLEM STATEMENT

Poiseuille flow is employed to test the boundary methods in the LBM for the Dirichlet velocity condition. Figure 1 shows 2D incompressible fluid flow between two parallel flat plates. The channel’s height and length are \( h \) and \( l \), respectively. Fully developed flow can be reached at the channel outlet since \( l \) is greater than \( 10h \).

This problem is governed by the following equations:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0 ,
\]

\[
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho uu)}{\partial x} + \frac{\partial (\rho vu)}{\partial y} = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) ,
\]

\[
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho uv)}{\partial x} + \frac{\partial (\rho vv)}{\partial y} = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) ,
\]

which are subject to the following boundary conditions:

\[
x = 0: \quad u = u(y) \quad v = 0 ,
\]

\[
x = l: \quad \partial u / \partial x = 0 \quad v = 0 ,
\]

\[
y = h / 2: \quad u = 0 \quad v = 0 ,
\]

\[
y = -h / 2: \quad u = 0 \quad v = 0 .
\]

In addition, the Reynolds number is defined using the maximum velocity \( u_{\text{max}} \) in the channel:

\[
\text{Re} = \frac{u_{\text{max}} h}{\nu} .
\]
3. LATTICE BOLTZMANN METHOD

The statistical behavior of fluid flow can be expressed by the following Boltzmann equation (Chen and Doolen, 1998):

\[
\frac{\partial f}{\partial t} + \mathbf{c} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{e}} = \Omega(f)_{\text{collision}},
\]

where \( f \) and \( \Omega \) are the density distribution and collision operator, respectively. The lattice Boltzmann method is executed on a regular grid. For a 2D problem, the density distributions at each computing node have nine directions to move to the nearby nodes, which is shown in Fig. 2. This is referred to as the D2Q9 model in the LBM (Chen and Doolen, 1998).

The velocities at each node are

\[
\begin{align*}
\mathbf{e}_a &= \begin{cases}
(0, 0) & a = 1 \\
(c(- \cos \frac{a\pi}{2}, - \sin \frac{a\pi}{2}) & a = 2, 3, 4, 5 \\
 \sqrt{2}c(- \cos \frac{(2a + 1)\pi}{4}, - \sin \frac{(2a + 1)\pi}{4}) & a = 6, 7, 8, 9
\end{cases}
\end{align*}
\]

where \( c \) is the lattice speed. Then Eq. (9) can be differenced in these nine directions as follows:

\[
f_a(x + e_a \Delta t, t + \Delta t) - f_a(x, t) = \Omega_a(f)_{\text{collision}} \quad a = 1, 2, \ldots, 9,
\]

where \( \Delta t \) is the discrete time step. One LBM iteration includes two steps: collision and streaming. The collision step is local to each node:

\[
f^*_a(x, t) = f_a(x, t) + \Omega_a(f)_{\text{collision}} \quad a = 1, 2, \ldots, 9,
\]

where \( f^* \) is the post-collision density distribution. Many methods exist in the LBM to simplify the collision term. The algorithms can be single-relaxation time model (SRT), double-relaxation time model, or multiple relaxation time model. The double-relaxation and multiple-relaxation time models have better numerical sta-
bility than SRT. On the other hand, SRT has valid advantages in a simple model setting. SRT is employed since numerical stability is not challenging for all the test cases in this article. This model simplifies the collision operator by the following equation:

$$
\Omega_a (f)_{\text{collision}} = -\frac{1}{\tau} \left( f_a - f_a^{eq} \right) \quad a = 1, 2, \ldots, 9,
$$

(13)

where $$\tau$$ is the single relaxation time and $$f_a^{eq}$$ is the direction equilibrium distribution

$$
f_a^{eq} = \rho \omega_a \left( 1 + \frac{1}{c_s^2} e_a \cdot V + \frac{1}{2c_s^4} Q_a : VV \right) \quad a = 1, 2, \ldots, 9,
$$

(14)

where $$c_s$$ is the speed of sound which is equal to $$c/\sqrt{5}$$. The scalar lattice weights $$\omega_a$$ and tensors $$Q_a$$ are defined as

$$
\omega_a = \begin{cases} 
\frac{4}{9} & a = 1 \\
\frac{1}{9} & a = 2, 3, 4, 5 \\
\frac{1}{36} & a = 6, 7, 8, 9 
\end{cases}
$$

(15)

$$
Q_a = e_a e_a - c_s^2 I \quad a = 1, 2, \ldots, 9,
$$

(16)

where $$I$$ is the unit tensor. The local collision step can be fulfilled by the settings given above. The streaming step follows the collision step and it takes the post-collision distributions to the nearby nodes:

$$
f_a \left( x + e_a \Delta t, t + \Delta t \right) = f_a^* \left( x, t \right) \quad a = 1, 2, \ldots, 9.
$$

(17)

The macroscopic variables can be obtained by moments of the density distributions. The quantities $$\rho$$, $$\rho V$$, and $$\Pi$$ correspond to the density distribution momentums of 0, 1, and 2:

$$
\rho = \sum_{a=1}^{9} f_a,
$$

(18)

$$
\rho V = \sum_{a=1}^{9} e_a f_a,
$$

(19)

$$
\Pi = \sum_{a=1}^{9} Q_a f_a,
$$

(20)

$$
p = \rho c_s^2.
$$

(21)

Applying the following Chapman–Enskog expansion equations:

$$
\frac{\partial}{\partial r} = K \frac{\partial}{\partial \eta},
$$

(22)
\[
\frac{\partial}{\partial t} = K \frac{\partial}{\partial t_1} + K \frac{\partial}{\partial t_2}, \quad (23)
\]

\[
f_a = f_a^0 + Kf_a^1 + K^2 f_a^2 \quad (24)
\]

to Eq. (11), the macroscopic governing equations can be obtained from the LBM:

\[
\frac{\partial (\rho V)}{\partial t} + \nabla \cdot (\rho V) = \nabla \cdot \left[ \rho \Delta t \left( \tau - \frac{1}{2} \right) c_s^2 \left( \nabla V + (\nabla V)^T - \frac{1}{c_s^2} \nabla \cdot (\rho VV) \right) \right]. \quad (26)
\]

To reach the Navier–Stokes equations, the relaxation time \( \tau \) is related to \( \nu \) by

\[
\Delta t \left( \tau - \frac{1}{2} \right) c_s^2 = \nu. \quad (27)
\]

The product \( K^2 f_a^2 \) in Eq. (24) shows no effect in this process. Equation (26) differs from the momentum equation due to presence of the term \( \nabla \cdot \left[ \rho \left( -\frac{\nu}{c_s^2} \nabla \cdot (\rho VV) \right) \right] \). It can be neglected when the Mach number is low (less than 0.3), which is the case in consideration. In the multiscale analysis process, the following equations are also obtained:

\[
\begin{cases}
f_a^0 = f_a^{eq} \\
Kf_a^1 = -\frac{\rho \tau \omega_a}{c_s^2} Q_a : \nabla V \\
\end{cases} \quad a = 1, 2, \ldots, 9. \quad (28)
\]

Neglecting the high-order term effect, the density distributions are approximated as

\[
f_a = f_a^{eq} - \frac{\rho \tau \omega_a}{c_s^2} Q_a : \nabla V \quad a = 1, 2, \ldots, 9. \quad (29)
\]

This equation plays important roles in many boundary conditions.

4. IMPLEMENTATION OF BOUNDARY CONDITIONS

The density distributions in some directions are unknown on the boundary before the collision step. Figure 3 shows the left boundary in a 2D domain.

The density distributions \( f_2, f_6, \) and \( f_9 \), shown by dark vectors, are unknown after the streaming step. The LBM boundary conditions are designed to find these unknown density distributions. Some boundary conditions substitute the unknown density distributions and keep the known ones. In contrast, some methods replace all the density distributions on boundaries before the collision step. The boundary conditions also differ from each other by the relation with the other near the nodes. Some methods obtain the boundary distributions based on the local computation node information only, while others need the nearby nodes information to recover the boundary distributions. This article compares three common methods [the Zou–He
method (Zou and He, 1997), the finite difference velocity gradient method (Skordos, 1993), and the regularized method (Latt, 2007) for the Dirichlet velocity boundary conditions. Table 1 summarizes the categories for these three methods. Boundary velocities are known in the Dirichlet condition. Three boundary conditions are shown in Fig. 3 for the Dirichlet velocity condition based on the left boundary.

4.1 Zou–He Boundary (BC1)

This method recovers the missing density distributions based on the local information only (Zou and He, 1997). In this 2D problem, Eqs. (18) and (19) become

\[
\begin{align*}
&f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + f_9 = \rho \\
&f_2 + f_6 + f_9 - f_4 - f_7 - f_8 = \rho u \\
&f_3 + f_6 + f_7 - f_5 - f_8 - f_9 = \rho v
\end{align*}
\]

where \(u\) and \(v\) are known from the Dirichlet velocity condition. Four independent unknowns (\(\rho, f_2, f_6,\) and \(f_9\)) in three equations indicate that one degree of freedom is left. Meanwhile, the density distribution can be obtained directly from Eq. (30):

**TABLE 1:** The categories of boundary conditions

<table>
<thead>
<tr>
<th>Method</th>
<th>Replace All (f_a)</th>
<th>Local</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zou–He (BC1)</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Finite difference (BC2)</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Regularized (BC3)</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>
\[ \rho = \frac{f_1 + f_3 + f_5 + 2 \times (f_4 + f_7 + f_8)}{1 + u}. \] (31)

At the four corner nodes, there is not enough information for obtaining the density. Extrapolating these corner nodes' densities (Latt and Chopard, 2008) from the neighboring computing nodes is employed for all three boundary conditions in this article.

In the Zou–He boundary condition, the nonequilibrium parts are assumed to be the same in directions 2 and 4:

\[ f_2 = f_2^{eq} + f_4 - f_4^{eq} \] (32)

and \( \frac{\tau_{02}}{c_s^2} \mathbf{Q}_2 : \rho \nabla \mathbf{V} \) is equal to \( \frac{\tau_{04}}{c_s^2} \mathbf{Q}_4 : \rho \nabla \mathbf{V} \) due to the symmetry of \( \mathbf{Q} \); therefore this assumption is reasonable regarding Eq. (27). Then the other two missing density distributions are

\[
\begin{align*}
 f_6 &= (f_4 + f_5 + 2 \times f_8 - f_2 - f_3 + \rho u + \rho v)/2 \\
 f_9 &= (f_3 + f_4 + 2 \times f_7 - f_2 - f_5 + \rho u - \rho v)/2.
\end{align*}
\] (33)

### 4.2 Finite Difference Velocity Gradient Method (BC2)

This method is based on the approximation in Eq. (29) with a slight difference (Skordos, 1993) where:

\[ f_a = f_a^{eq} - \frac{\tau_{0a}}{c_s^2} \mathbf{Q}_a : \nabla (\rho \mathbf{V}) \quad a = 1, 2, \ldots, 9, \] (34)

is the double dot product. Differently from Eq. (29), \( \rho \) is inside the operator \( \nabla \).

This setting includes the compressible effect on the strain rate, which is negligible for the incompressible fluid flow in this article. The boundary density is obtained from Eq. (30) and then \( \nabla (\rho \mathbf{V}) \) is obtained by the following equation:

\[ \nabla \mathbf{V} = \begin{bmatrix} \partial (\rho u)/\partial x, & \partial (\rho v)/\partial x \\ \partial (\rho u)/\partial y, & \partial (\rho v)/\partial y \end{bmatrix}. \] (35)

To maintain the second-order accuracy, three point midpoint and endpoint derivative laws are employed for the operators in Eq. (35). For the function \( g \) taken as an example, three computing nodes in one direction are shown in Fig. 4. Its first derivative functions \( g' \) at different locations can be obtained from the following equation:

\[
\begin{align*}
 g'(i) &= \frac{1}{2\Delta x} \left[ -3 \times g(i) + 4 \times g(i + 1) - g(i + 2) \right] \\
 g'(i + 1) &= \frac{1}{2\Delta x} \left[ -g(i) + g(i + 2) \right] \\
 g'(i + 2) &= \frac{1}{2\Delta x} \left[ g(i) - 4 \times g(i + 1) + 3 \times g(i + 2) \right].
\end{align*}
\] (36)

\( \partial (\rho u)/\partial x, \partial (\rho v)/\partial x, \partial (\rho u)/\partial y, \) and \( \partial (\rho v)/\partial y \) in Eq. (35) can be obtained from Eq. (36) for all the boundaries including the corner nodes. Then all the boundary density distributions can be obtained from Eq. (34).
4.3 Regularized Method (BC3)

Similar to the finite velocity gradient method, the regularized method (Latt, 2007) also replaces all the boundary density distributions with the aid of Eq. (29). Instead of calculating $\nabla V$ from the nearby nodes information, the regularized method fulfills this step using the local information. The strain rate tensor $S$ is defined as

$$\frac{1}{2} \left[ \nabla V + (\nabla V)^T \right].$$

Due to the symmetry of $Q_a$ and $S$, Eq. (29) becomes

$$f_a = f_a^{eq} + \frac{\omega_a}{2c_s^2} Q_a : \Pi^{(1)} \quad a = 1, 2, \ldots, 9,$$

where the tensor $\Pi^{(1)}$ is defined as

$$\Pi^{(1)} = -2c_s^2 \tau p S.$$

After the solution of the boundary density $\rho$ from Eq. (30), $f_a^{eq}$ can be obtained from Eq. (14). The quantity $\Pi^{(1)}$ is approximated by the following equation:

$$\Pi^{(1)} = \Pi^{(neq)} = \sum_{a=1}^{9} Q_a f_a^{neq}.$$  \hspace{1cm} (39)

Not all $f_a^{neq}$ are known after the streaming process. For example, $f_2^{neq}$, $f_6^{neq}$, and $f_9^{neq}$ are unknowns for the left boundary. They are approximated to equal those in the opposite directions:

$$f_2^{neq} = f_4^{neq}, \quad f_6^{neq} = f_8^{neq}, \quad f_9^{neq} = f_7^{neq}.$$  \hspace{1cm} (40)

Then the regularized method if fulfilled by the above-given equations.

5. RESULTS AND DISCUSSION

To test the three methods for the Dirichlet velocity boundary conditions, Poiseuille flows in Section 2 are solved with the aid of the LBM. The number Re in the test cases is equal to 5, 10, 25, and 50 to limit the compressible effects. After grid size independent test, $30 \times 500$ grids are employed for all the test cases.

An analytical solution exists in this problem once full development is reached (Latt and Chopard, 2008):

$$u(y) = u_{max} \left[ 1 - \left( \frac{2y}{h} \right)^2 \right],$$  \hspace{1cm} (41)

where $u_{max}$ is the maximum velocity and then $u$ is quadratic for the location $y$. The quantity $p_x$ is the average pressure along $y$. It is linear relative to $x$ according to the analytical solution. Assuming $\Delta p$ to be the pressure difference between $p_x$ and $p_{out}$, $\Delta p$ is also linear relative to $x$. 

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Each has same $\tau$ in Eq. (27) by adapting $u_0$. The LBM simulation is carried on until the steady-state condition is reached:

$$\max \left| \frac{u_t - u_{t-1}}{u_t} \right| \leq 10^{-10}. \quad (42)$$

Numerical solutions are then compared with analytical results for validation. Figure 5 shows the comparison of velocities between the LBM and analytical results for $Re = 5$. These two quadratic results agree with each other well. Figure 6 shows the comparison of pressure differences. They are all linear relative to $x$ and agree with each other well. Therefore, numerical results with different boundary conditions agree with the analytical results well in this case.

Figures 7 and 8 show the velocity and pressure comparisons for $Re = 10$. The velocities are still quadratic relative to $y$ and agree with the analytical results well. The maximum velocity $u_{\text{max}}$ increases from 0.005 to 0.01 as $Re$ increases from 5 to 10. All three boundary conditions also lead to the same $\Delta p$ results which are
FIG. 6: Pressure distribution, Re = 5

FIG. 7: Velocity distribution, Re = 10
linear relative to \( x \). The slope is also twice as that for the case of \( Re = 5 \). Therefore, the LBM with different boundary conditions can lead to the results agreeing with analytical results well.

The test cases are also carried out for Reynolds numbers equal to 25 and 50. Figures 9–12 compare the velocity and pressure for these two cases. Their analytical results are also satisfied. Therefore, the LBM with different boundary conditions can lead to the results that agree with analytical results for all the test cases.

It is necessary to point out that these three boundary conditions all violate the local mass balance, which does not relate to mass balance directly. The parameter \( MB \) is employed to evaluate different boundary conditions’ accuracies:

\[
MB = \max \left[ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} \right].
\] (43)

It can be reached based on the macroscale parameters with the derivatives obtained from Eq. (36). This common convergence criterion in the CFD is not used widely in the LBM. This parameter grows with the velocity. To eliminate this effect, we use another error parameter:

\[
\text{error} = \frac{MB}{u_{\text{max}}}. \tag{44}
\]

The error tendency with \( Re \) for different boundary conditions is shown in Fig. 13.

Numerical methods fit mass balance better with this error decreasing. It changes from \( 10^{-5} \) to \( 10^{-4} \) for the BC1. On the other hand, BC2 and BC3 have the errors around \( 10^{-3} \). The BC1 error is less than the other boundary conditions' results by 10 times, at least. The BC1 results are much better than the other cases from the mass balance point of view.

6. CONCLUSIONS

Three boundary conditions (the Zou–He method, finite difference velocity gradient method, and the regularized method) are tested and compared for the Dirichlet velocity condition. Poiseuille flows with different
FIG. 9: Velocity distribution, Re = 25

FIG. 10: Pressure distribution, Re = 25
FIG. 11: Velocity distribution, Re = 50

FIG. 12: Pressure distribution, Re = 50
Reynolds numbers are solved for validation. The LBM velocity and pressure results with different boundary conditions agree with the analytical solutions well for all the test cases. Meanwhile, the LBM with the Zou–He boundary condition leads to the results fitting mass balance best; therefore, the local mass balance has no relation with the macroscopic mass balance in the LBM boundary. The Zou–He boundary condition is superior in the LBM for this kind of problem relating to the macroscopic mass balance.

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